Phase II. The tableau for the original problem (in standard form) is:

	a_1	a_2	a_3	a_4	b
	1	1	1	0	4
	5	3	0	−1	8
c^T	−3	−5	0	0	0

As the initial revised tableau for phase II, we take the final revised tableau from phase I. We then compute

$$\begin{aligned}\lambda^T &= c_B^T B^{-1} = [0,-3]\begin{bmatrix}1 & -\frac{1}{5}\\ 0 & \frac{1}{5}\end{bmatrix} = \left[0,-\frac{3}{5}\right],\\ r_D^T &= c_D^T - \lambda^T D = [-5,0] - \left[0,-\frac{3}{5}\right]\begin{bmatrix}1 & 0\\ 3 & -1\end{bmatrix} = \left[-\frac{16}{5},-\frac{3}{5}\right] = [r_2, r_4].\end{aligned}$$

We bring a_2 into the basis, and compute $y_2 = B^{-1}a_2$ to get:

Variable	B^{-1}		y_0	y_2
x_3	1	$-\frac{1}{5}$	$\frac{12}{5}$	$\frac{2}{5}$
x_1	0	$\frac{1}{5}$	$\frac{8}{5}$	$\frac{3}{5}$

In this case, we get $p = 2$. We update this tableau by pivoting about the 2nd element of the last column to get

Variable	B^{-1}		y_0
x_3	1	$-\frac{1}{3}$	$\frac{4}{3}$
x_2	0	$\frac{1}{3}$	$\frac{8}{3}$

We compute

$$\begin{aligned}\lambda^T &= c_B^T B^{-1} = [0,-5]\begin{bmatrix}1 & -\frac{1}{3}\\ 0 & \frac{1}{3}\end{bmatrix} = \left[0,-\frac{5}{3}\right],\\ r_D^T &= c_D^T - \lambda^T D = [-3,0] - \left[0,-\frac{5}{3}\right]\begin{bmatrix}1 & 0\\ 5 & -1\end{bmatrix} = \left[\frac{16}{3},-\frac{5}{3}\right] = [r_1, r_4].\end{aligned}$$

We now bring a_4 into the basis:

Variable	B^{-1}		y_0	y_4
x_3	1	$-\frac{1}{3}$	$\frac{4}{3}$	$\frac{1}{3}$
x_2	0	$\frac{1}{3}$	$\frac{8}{3}$	$-\frac{1}{3}$

We update the tableau to obtain:

Variable	B^{-1}		y_0
x_4	3	−1	4
x_2	1	0	4

We compute

$$\begin{aligned}\lambda^T &= c_B^T B^{-1} = [0,-5]\begin{bmatrix}3 & -1\\ 1 & 0\end{bmatrix} = [-5,0],\\ r_D^T &= c_D^T - \lambda^T D = [-3,0] - [-5,0]\begin{bmatrix}1 & 1\\ 5 & 0\end{bmatrix} = [2,5] = [r_1, r_3].\end{aligned}$$

The reduced cost coefficients are all positive. Hence, $[0,4,0,4]^T$ is optimal. The optimal solution to the original problem is $[0,4]^T$.

EXERCISES

16.1 Consider the following standard form LP problem:

$$\begin{aligned}\text{minimize}\quad & 2x_1 - x_2 - x_3\\ \text{subject to}\quad & 3x_1 + x_2 + x_4 = 4\\ & 6x_1 + 2x_2 + x_3 + x_4 = 5\\ & x_1, x_2, x_3, x_4 \ge 0.\end{aligned}$$

a. Write down the A, b, and c matrices/vectors for the problem.

b. Consider the basis consisting of the third and fourth columns of A, ordered according to $[a_4, a_3]$. Compute the canonical tableau corresponding to this basis.

c. Write down the basic feasible solution corresponding to the above basis, and its objective function value.

d. Write down the values of the reduced cost coefficients (for all the variables) corresponding to the above basis.

e. Is the basic feasible solution in part c an optimal feasible solution? If yes, explain why. If not, determine which element of the canonical tableau to pivot about so that the new basic feasible solution will have a lower objective function value.

f. Suppose we apply the two-phase method to the problem, and at the end of phase I, the tableau for the artificial problem is

$$\begin{array}{ccccccc}0 & 0 & -1 & 1 & 2 & -1 & 3\\ 1 & \frac{1}{3} & \frac{1}{3} & 0 & -\frac{1}{3} & \frac{1}{3} & \frac{1}{3}\\ 0 & 0 & 0 & 0 & 1 & 1 & 0\end{array}$$

Does the original problem have a basic feasible solution? Explain.

g. From the final tableau for phase I in part f, find the initial canonical tableau for phase II.

16.2 Use the simplex method to solve the following linear program:

$$\begin{array}{ll} \text{maximize} & x_1 + x_2 + 3x_3 \\ \text{subject to} & x_1 + x_3 = 1 \\ & x_2 + x_3 = 2 \\ & x_1, x_2, x_3 \geq 0. \end{array}$$

16.3 Consider the linear program:

$$\begin{array}{ll} \text{maximize} & 2x_1 + x_2 \\ \text{subject to} & 0 \leq x_1 \leq 5 \\ & 0 \leq x_2 \leq 7 \\ & x_1 + x_2 \leq 9. \end{array}$$

Convert the problem to standard form and solve it using the simplex method.

16.4 Consider a standard form linear programming problem (with the usual $\boldsymbol{A}$, $\boldsymbol{b}$, and $\boldsymbol{c}$). Suppose that it has the following canonical tableau:

$$\begin{array}{cccccc} 0 & 1 & 0 & 1 & -1 & 5 \\ 1 & 2 & 0 & 0 & -2 & 6 \\ 0 & 3 & 1 & 0 & -3 & 7 \\ 0 & 4 & 0 & 0 & -4 & 8 \end{array}$$

a. Find the basic feasible solution corresponding to the above canonical tableau, and the corresponding value of the objective function.

b. Find all the reduced cost coefficient values associated with the above canonical tableau.

c. Does the given linear programming problem have feasible solutions with arbitrarily negative objective function values?

d. Suppose column $\boldsymbol{a}_2$ enters the basis. Find the canonical tableau for the new basis.

e. Find a feasible solution with objective function value equal to -100.

f. Find a basis for the nullspace of $\boldsymbol{A}$.

16.5 Consider the problem:

$$\begin{array}{ll} \text{maximize} & -x_1 - 2x_2 \\ \text{subject to} & x_1 \geq 0 \\ & x_2 \geq 1. \end{array}$$

a. Convert the problem into a standard form linear programming problem.

b. Use the two-phase simplex method to compute the solution to the above given problem, and the value of the objective function at the optimal solution of the given problem.

16.6 Consider the linear programming problem:

$$\begin{array}{ll} \text{minimize} & -x_1 \\ \text{subject to} & x_1 - x_2 = 1 \\ & x_1, x_2 \geq 0. \end{array}$$

a. Write down the basic feasible solution for x_1 as a basic variable.

b. Compute the canonical augmented matrix corresponding to the basis in part a.

c. If we apply the simplex algorithm to this problem, under what circumstance does it terminate? (In other words, which stopping criterion in the simplex algorithm is satisfied?)

d. Show that in this problem, the objective function can take arbitrarily negative values over the constraint set.

16.7 Find the solution and the value of the optimal cost for the following problem using the revised simplex method:

$$\begin{array}{ll} \text{minimize} & x_1 + x_2 \\ \text{subject to} & x_1 + 2x_2 \geq 3 \\ & 2x_1 + x_2 \geq 3 \\ & x_1, x_2 \geq 0. \end{array}$$

Hint: Start with x_1 and x_2 as basic variables.

16.8 Solve the following linear programs using the revised simplex method:

a. maximize $-4x_1 - 3x_2$ subject to

$$\begin{array}{rcl} 5x_1 + x_2 & \geq & 11 \\ -2x_1 - x_2 & \leq & -8 \\ x_1 + 2x_2 & \geq & 7 \\ x_1, x_2 & \geq & 0. \end{array}$$

b. maximize $6x_1 + 4x_2 + 7x_3 + 5x_4$ subject to

$$\begin{aligned} x_1 + 2x_2 + x_3 + 2x_4 &\le 20 \\ 6x_1 + 5x_2 + 3x_3 + 2x_4 &\le 100 \\ 3x_1 + 4x_2 + 9x_3 + 12x_4 &\le 75 \\ x_1, x_2, x_3, x_4 &\ge 0. \end{aligned}$$

16.9 Consider a standard form linear programming problem, with

$$A = \begin{bmatrix} 0 & 2 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}, \quad c = \begin{bmatrix} 6 \\ c_2 \\ 4 \\ 5 \end{bmatrix}.$$

Suppose that we are told that the reduced cost coefficient vector corresponding to some basis is $r^T = [0, 1, 0, 0]$.

a. Find an optimal feasible solution to the given problem.

b. Find an optimal feasible solution to the dual of the given problem.

c. Find c_2.

16.10 Consider the linear programming problem:

$$\begin{aligned} \text{minimize} \quad & c_1x_1 + c_2x_2 \\ \text{subject to} \quad & 2x_1 + x_2 = 2 \\ & x_1, x_2 \ge 0, \end{aligned}$$

where $c_1, c_2 \in \mathbb{R}$. Suppose that the problem has an optimal feasible solution that is not basic.

a. Find all basic feasible solutions.

b. Find all possible values of c_1 and c_2.

c. At each basic feasible solution, compute the reduced cost coefficients for all nonbasic variables.

16.11 Suppose we apply the Simplex method to a given linear programming problem, and obtain the following canonical tableau:

$$\begin{matrix} 0 & \beta & 0 & 1 & 4 \\ 1 & \gamma & 0 & 0 & 5 \\ 0 & -3 & 1 & 0 & 6 \\ 0 & 2-\alpha & 0 & 0 & \delta \end{matrix}$$

For each of the following conditions, find the set of all parameter values $\alpha, \beta, \gamma, \delta$ that satisfy the given condition.

a. The problem has no solution because the objective function values are unbounded.

b. The current basic feasible solution is optimal, and the corresponding objective function value is 7.

c. The current basic feasible solution is not optimal, and the objective function value strictly decreases if we remove the first column of A from the basis.

16.12 Consider the following linear programming problem (attributed to Beale—see [28, p. 43]):

$$\begin{aligned} \text{minimize} \quad & -\frac{3}{4}x_4 + 20x_5 - \frac{1}{2}x_6 + 6x_7 \\ \text{subject to} \quad & x_1 + \frac{1}{4}x_4 - 8x_5 - x_6 + 9x_7 = 0 \\ & x_2 + \frac{1}{2}x_4 - 12x_5 - \frac{1}{2}x_6 + 3x_7 = 0 \\ & x_3 + x_6 = 1 \\ & x_1, \dots, x_7 \ge 0. \end{aligned}$$

a. Apply the simplex algorithm to the problem using the rule that q is the index corresponding to the most negative r_q. (As usual, if more than one index i minimizes y_{i0}/y_{iq}, let p be the smallest such index.) Start with x_1, x_2, and x_3 as initial basic variables. Notice that cycling occurs.

b. Repeat part a using *Bland's rule* for choosing q and p:

$$\begin{aligned} q &= \min\{i : r_i < 0\} \\ p &= \min\{j : y_{j0}/y_{jq} = \min_i\{y_{i0}/y_{iq} : y_{iq} > 0\}\}. \end{aligned}$$

Note that Bland's rule for choosing p corresponds to our usual rule that if more than one index i minimizes y_{i0}/y_{iq}, we let p be the smallest such index.

16.13 Write a simple MATLAB function that implements the simplex algorithm. The inputs are c, A, b, and v, where v is the vector of indices of basic columns. Assume that the augmented matrix $[A, b]$ is already in canonical form, that is, the v_ith column of A is $[0, \dots, 1, \dots, 0]^T$, where 1 occurs in the ith position. The function should output the final solution and the vector of indices of basic columns. Test the MATLAB function on the problem in Example 16.2.