For this solution, the objective function value is -30, which is an improvement relative to the objective function value at the old extreme point.

We now apply the same procedure as above to move to another adjacent extreme point, which hopefully further decreases the value of the objective function. This time, we choose $\boldsymbol{a}_2$ to enter the new basis. We have

$$\boldsymbol{a}_2 = \frac{1}{2}\boldsymbol{a}_1 + \frac{9}{2}\boldsymbol{a}_3 + \frac{1}{2}\boldsymbol{a}_5,$$

and

$$\left(10 - \frac{1}{2}\varepsilon_2\right)\boldsymbol{a}_1 + \varepsilon_2\boldsymbol{a}_2 + \left(30 - \frac{9}{2}\varepsilon_2\right)\boldsymbol{a}_3 + \left(2 - \frac{1}{2}\varepsilon_2\right)\boldsymbol{a}_5 = \boldsymbol{b}.$$

Substituting $\varepsilon_2 = 4$, we obtain

$$8\boldsymbol{a}_1 + 4\boldsymbol{a}_2 + 12\boldsymbol{a}_3 = \boldsymbol{b}.$$

The solution is $[8, 4, 12, 0, 0]^T$ and the corresponding value of the objective function is -44, which is smaller than the value at the previous extreme point. To complete the example, we repeat the procedure once more. This time, we select $\boldsymbol{a}_4$ and express it as a combination of the vectors in the previous basis, $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, and $\boldsymbol{a}_3$:

$$\boldsymbol{a}_4 = \boldsymbol{a}_1 - \boldsymbol{a}_2 + 4\boldsymbol{a}_3,$$

and hence

$$(8 - \varepsilon_3)\boldsymbol{a}_1 + (4 + \varepsilon_3)\boldsymbol{a}_2 + (12 - 4\varepsilon_3)\boldsymbol{a}_3 + \varepsilon_3\boldsymbol{a}_4 = \boldsymbol{b}.$$

The largest permissible value for ε_3 is 3. The corresponding basic feasible solution is $[5, 7, 0, 3, 0]^T$, with an objective function value of -50. The solution $[5, 7, 0, 3, 0]^T$ turns out to be an optimal solution to our problem in standard form. Hence, the solution to the original problem is $[5, 7]^T$, which we can easily obtain graphically (see Figure 15.11). ■

The technique used in the above example for moving from one extreme point to an adjacent extreme point is also used in the simplex method for solving LP problems. The simplex method is essentially a refined method of performing these manipulations.

15.1 15.7
15.3 15.8

EXERCISES

15.1 Convert the following linear programming problem to *standard form:*

$$\begin{array}{ll} \text{maximize} & 2x_1 + x_2 \\ \text{subject to} & 0 \le x_1 \le 2 \\ & x_1 + x_2 \le 3 \\ & x_1 + 2x_2 \le 5 \\ & x_2 \ge 0. \end{array}$$

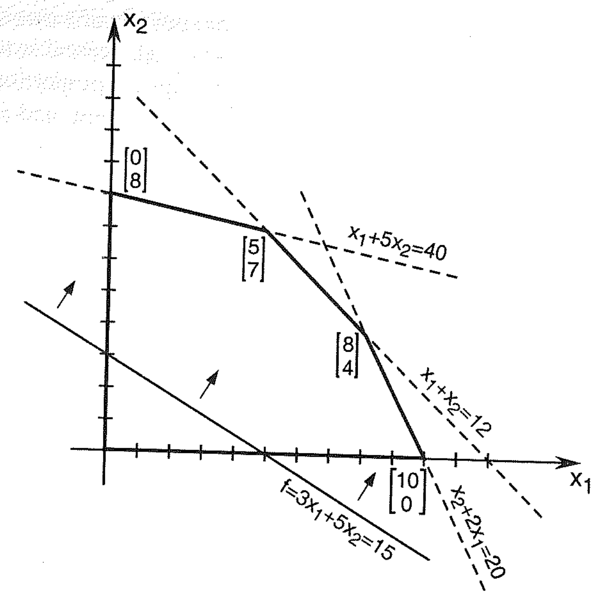

Figure 15.11 A graphical solution to the LP problem in Example 15.14

15.2 Consider a discrete-time linear system $x_{k+1} = ax_k + bu_k$, where u_k is the input at time k, x_k is the output at time k, and $a, b \in \mathbb{R}$ are system parameters. Given an initial condition $x_0 = 1$, consider the problem of minimizing the output x_2 at time 2 subject to the constraint that $|u_i| \le 1$, $i = 0, 1$.

Formulate the problem as a linear programming problem, and convert it into standard form.

15.3 Consider the optimization problem

$$\begin{array}{ll} \text{minimize} & c_1|x_1| + c_2|x_2| + \cdots + c_n|x_n| \\ \text{subject to} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \end{array}$$

where $c_i \ne 0$, $i = 1, \ldots, n$. Convert the above problem into an equivalent standard form linear programming problem.

Hint: Given any $x \in \mathbb{R}$, we can find unique numbers $x^+, x^- \in \mathbb{R}$, $x^+, x^- \ge 0$, such that $|x| = x^+ + x^-$ and $x = x^+ - x^-$.

15.4 Does every linear programming problem in standard form have a nonempty feasible set? If yes, prove. If no, give a specific example.

Does every linear programming problem in standard form (assuming a nonempty feasible set) have an optimal solution? If yes, prove. If no, give a specific example.

15.5 A cereal manufacturer wishes to produce 1000 pounds of a cereal that contains exactly 10% fiber, 2% fat, and 5% sugar (by weight). The cereal is to be produced by combining four items of raw food material in appropriate proportions. These four items have certain combinations of fiber, fat, and sugar content, and are available at various prices per pound, as shown below:

Item	1	2	3	4
% fiber	3	8	16	4
% fat	6	46	9	9
% sugar	20	5	4	0
Price/lb.	2	4	1	2

The manufacturer wishes to find the amounts of each of the above items to be used to produce the cereal in the least expensive way. Formulate the problem as a linear programming problem. What can you say about the existence of a solution to this problem?

15.6 Suppose a wireless broadcast system has n transmitters. Transmitter j broadcasts at a power of $p_j \geq 0$. There are m locations where the broadcast is to be received. The "path gain" from transmitter j to location i is $g_{i,j}$; that is, the power of the signal transmitted from transmitter j received at location i is $g_{i,j}p_j$. The total received power at location i is the sum of the received powers from all the transmitters.

Formulate the problem of finding the minimum sum of the transmit powers subject to the requirement that the received power at each location is at least P.

15.7 Consider the system of equations:

$$\begin{bmatrix} 2 & -1 & 2 & -1 & 3 \\ 1 & 2 & 3 & 1 & 0 \\ 1 & 0 & -2 & 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 14 \\ 5 \\ -10 \end{bmatrix}.$$

Check if the system has basic solutions. If yes, find all basic solutions.

15.8 Solve the following linear program graphically:

$$\begin{array}{ll} \text{maximize} & 2x_1 + 5x_2 \\ \text{subject to} & 0 \leq x_1 \leq 4 \\ & 0 \leq x_2 \leq 6 \\ & x_1 + x_2 \leq 8. \end{array}$$

15.9 The optimization toolbox in MATLAB provides a function, `linprog`, for solving linear programming problems. Use the function `linprog` to solve the problem in Example 15.5. Use the initial condition $\mathbf{0}$.