Note that, because $\mu^* = 0$, the active constraint $g(\boldsymbol{x}^*) = 0$ does not enter the computation of $\tilde{T}(\boldsymbol{x}^*, \mu^*)$. Note also that in this case, $T(\boldsymbol{x}^*) = \{\boldsymbol{0}\}$. We have

$$\tilde{T}(\boldsymbol{x}^*, \mu^*) = \{\boldsymbol{y} : [-1, 1]\boldsymbol{y} = 0\} = \{[a, a]^T : a \in \mathbb{R}\}.$$

We then check for positive definiteness of $\boldsymbol{L}(\boldsymbol{x}^*, \lambda^*, \mu^*)$ on $\tilde{T}(\boldsymbol{x}^*, \mu^*)$. We have

$$\boldsymbol{y}^T \boldsymbol{L}(\boldsymbol{x}^*, \lambda^*, \mu^*)\boldsymbol{y} = [a, a] \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ a \end{bmatrix} = 2a^2.$$

Thus, $\boldsymbol{L}(\boldsymbol{x}^*, \lambda^*, \mu^*)$ is positive definite on $\tilde{T}(\boldsymbol{x}^*, \mu^*)$. Observe that $\boldsymbol{L}(\boldsymbol{x}^*, \lambda^*, \mu^*)$ is, in fact, only positive semidefinite on $\mathbb{R}^2$.

By the second-order sufficient conditions, we conclude that $\boldsymbol{x}^* = [1/2, 3/2]^T$ is a strict local minimizer. ■

EXERCISES

20.1 Find local extremizers for

a. $x_1^2 + x_2^2 - 2x_1 - 10x_2 + 26$ subject to $\frac{1}{5}x_2 - x_1^2 \leq 0$, $5x_1 + \frac{1}{2}x_2 \leq 5$;

b. $x_1^2 + x_2^2$ subject to $x_1 \geq 0$, $x_2 \geq 0$, $x_1 + x_2 \geq 5$;

c. $x_1^2 + 6x_1x_2 - 4x_1 - 2x_2$ subject to $x_1^2 + 2x_2 \leq 1$, $2x_1 - 2x_2 \leq 1$.

20.2 Find local minimizers for $x_1^2 + x_2^2$ subject to $x_1^2 + 2x_1x_2 + x_2^2 = 1$, $x_1^2 - x_2 \leq 0$.

20.3 Write down the Karush-Kuhn-Tucker condition for the optimization problem in Exercise 15.6.

20.4 Consider the problem

$$\begin{array}{ll} \text{minimize} & x_2 - (x_1 - 2)^3 + 3 \\ \text{subject to} & x_2 \geq 1, \end{array}$$

where x_1 and x_2 are real variables. Answer each of the following questions, making sure that you give complete reasoning for your answers.

a. Write down the KKT condition for the problem, and find all points that satisfy the condition. Check whether or not each point is regular.

b. Determine whether or not the point(s) in part a satisfy the second-order necessary condition.

c. Determine whether or not the point(s) in part b satisfy the second-order sufficient condition.

20.5 Consider the problem

$$\begin{array}{ll} \text{minimize} & x_2 \\ \text{subject to} & x_2 \geq -(x_1 - 1)^2 + 3. \end{array}$$

a. Find all points satisfying the KKT condition for the problem.

b. For each point $\boldsymbol{x}^*$ in part a, find $T(\boldsymbol{x}^*)$, $N(\boldsymbol{x}^*)$, and $\tilde{T}(\boldsymbol{x}^*)$.

c. Find the subset of points from part a that satisfy the second-order necessary condition.

20.6 Consider the optimization problem

$$\begin{array}{ll} \text{minimize} & f(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{x} \in \Omega, \end{array}$$

where $f(\boldsymbol{x}) = x_1 x_2^2$, where $\boldsymbol{x} = [x_1, x_2]^T$, and $\Omega = \{\boldsymbol{x} \in \mathbb{R}^2 : x_1 = x_2,\ x_1 \geq 0\}$.

a. Find all points satisfying the KKT condition.

b. Do each of the points found in part a satisfy the second-order necessary condition?

c. Do each of the points found in part a satisfy the second-order sufficient condition?

20.7 Consider the problem

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}\|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}\|^2 \\ \text{subject to} & x_1 + \cdots + x_n = 1 \\ & x_1, \ldots, x_n \geq 0. \end{array}$$

a. Write down the KKT condition for the problem.

b. Define what it means for a feasible point $\boldsymbol{x}^*$ to be *regular* in this particular given problem. Are there any feasible points in this problem that are not regular? If yes, find them. If not, explain why not.

20.8 Let $g : \mathbb{R}^n \to \mathbb{R}$ and $\boldsymbol{x}_0 \in \mathbb{R}^n$ be given, where $g(\boldsymbol{x}_0) > 0$. Consider the problem

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}\|\boldsymbol{x} - \boldsymbol{x}_0\|^2 \\ \text{subject to} & g(\boldsymbol{x}) \leq 0. \end{array}$$

Suppose $\boldsymbol{x}^*$ is a solution to the problem, and $g \in C^1$. Use the KKT theorem to decide which of the following equations/inequalities hold:

i. $g(\boldsymbol{x}^*) < 0$

ii. $g(\boldsymbol{x}^*) = 0$

iii. $(\boldsymbol{x}^* - \boldsymbol{x}_0)^T \nabla g(\boldsymbol{x}^*) < 0$

iv. $(\boldsymbol{x}^* - \boldsymbol{x}_0)^T \nabla g(\boldsymbol{x}^*) = 0$

v. $(\boldsymbol{x}^* - \boldsymbol{x}_0)^T \nabla g(\boldsymbol{x}^*) > 0$.

20.9 Consider a square room, with corners located at $[0,0]^T$, $[0,2]^T$, $[2,0]^T$, and $[2,2]^T$ (in $\mathbb{R}^2$). We wish to find the point in the room that is closest to the point $[3,4]^T$.

a. Guess which point in the room is the closest point in the room to the point $[3,4]^T$.

b. Use the second-order sufficient conditions to prove that the point you have guessed is a strict local minimizer.

Hint: Minimizing the distance is the same as minimizing the square distance.

20.10 Consider the *quadratic programming* problem

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}\boldsymbol{x}^T \boldsymbol{Q}\boldsymbol{x} \\ \text{subject to} & \boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b}, \end{array}$$

where $\boldsymbol{Q} = \boldsymbol{Q}^T > 0$, $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, and $\boldsymbol{b} \ge \boldsymbol{0}$. Find all points satisfying the KKT condition.

20.11 Consider the problem

$$\begin{array}{ll} \text{minimize} & \boldsymbol{c}^T \boldsymbol{x} \\ \text{subject to} & \boldsymbol{A}\boldsymbol{x} \le \boldsymbol{0}, \end{array}$$

where $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, $m < n$, is of full rank. Use the KKT theorem to show that if there exists a solution, then the optimal objective function value is 0.

20.12 Consider a linear programming problem in standard form (see Chapter 15).

a. Write down the Karush-Kuhn-Tucker condition for the problem.

b. Use part a to show that if there exists an optimal feasible solution to the linear program, then there exists a feasible solution to the corresponding dual problem that achieves an objective function value that is the same as the optimal value of the primal (compare this with Theorem 17.1).

c. Use parts a and b to prove that if $\boldsymbol{x}^*$ is an optimal feasible solutions of the primal, then there exists a feasible solution $\boldsymbol{\lambda}^*$ to the dual such that $(\boldsymbol{c}^T - \boldsymbol{\lambda}^{*T}\boldsymbol{A})\boldsymbol{x}^* = 0$ (compare this with Theorem 17.3).

20.13 Consider the constraint set $S = \{\boldsymbol{x} : \boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{0}, \boldsymbol{g}(\boldsymbol{x}) \le \boldsymbol{0}\}$. Let $\boldsymbol{x}^* \in S$ be a regular local minimizer of f over S, and $J(\boldsymbol{x}^*)$ the index set of active inequality constraints. Show that $\boldsymbol{x}^*$ is also a regular local minimizer of f over the set $S' = \{\boldsymbol{x} : \boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{0}, g_j(\boldsymbol{x}) = 0, j \in J(\boldsymbol{x}^*)\}$.

20.14 Solve the following optimization problem using the second-order sufficient conditions:

$$\begin{array}{ll} \text{minimize} & x_1^2 + x_2^2 \\ \text{subject to} & x_1^2 - x_2 - 4 \le 0 \\ & x_2 - x_1 - 2 \le 0. \end{array}$$

See Figure 21.1 for a graphical illustration of the problem.

20.15 Solve the following optimization problem using the second-order sufficient conditions:

$$\begin{array}{ll} \text{minimize} & x_1^2 + x_2^2 \\ \text{subject to} & x_1 - x_2^2 - 4 \ge 0 \\ & x_1 - 10 \le 0. \end{array}$$

See Figure 21.2 for a graphical illustration of the problem.

20.16 Consider the problem

$$\begin{array}{ll} \text{minimize} & x_1^2 + x_2^2 \\ \text{subject to} & 4 - x_1 - x_2^2 \le 0 \\ & 3x_2 - x_1 \le 0 \\ & -3x_2 - x_1 \le 0. \end{array}$$

Figure 21.3 gives a graphical illustration of the problem. Deduce from the figure that the problem has two strict local minimizers, and use the second-order sufficient conditions to verify the graphical solutions.

20.17 Consider the problem:

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}\|\boldsymbol{x}\|^2 \\ \text{subject to} & \boldsymbol{a}^T \boldsymbol{x} = b \\ & \boldsymbol{x} \ge \boldsymbol{0}, \end{array}$$

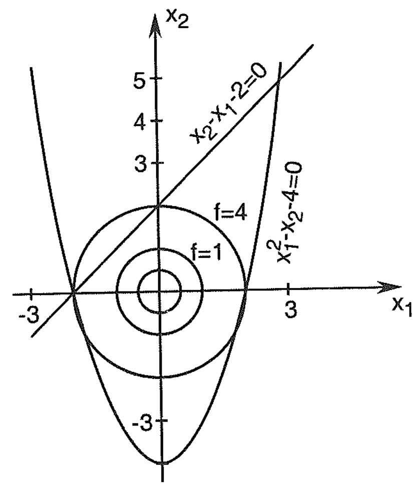

Figure 21.1 Situation where the constrained and the unconstrained minimizers are the same

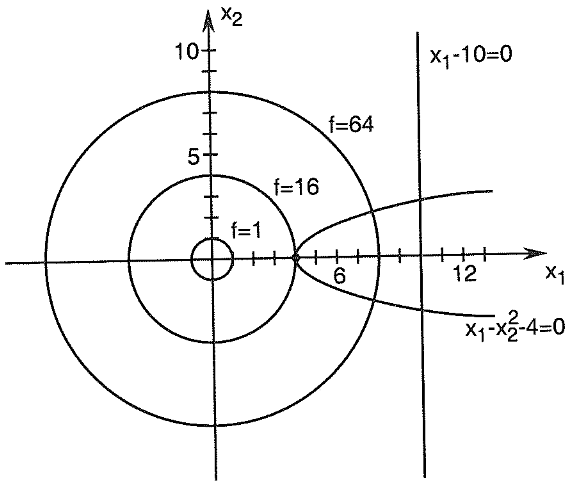

Figure 21.2 Situation where only one constraint is active

The problem is depicted in Figure 21.2. At the solution, only one constraint is active. If we had only known about this we could have handled this problem as a constrained optimization problem using the Lagrange multiplier method. ∎

Example 21.3 Consider the optimization problem

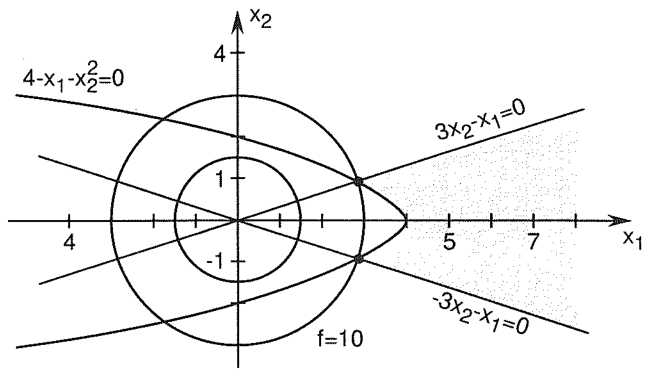

Figure 21.3 Situation where the constraints introduce local minimizers

$$3x_2 - x_1 \leq 0$$
$$-3x_2 - x_1 \leq 0.$$

The problem is depicted in Figure 21.3. This example illustrates the situation where the constraints introduce local minimizers, even though the objective function itself has only one unconstrained global minimizer. ∎

Some of the difficulties illustrated in the above examples can be eliminated if we restrict our problems to convex feasible regions. Admittedly, some important real-life problems do not fit into this framework. On the other hand, it is possible to give results of a *global* nature for this class of optimization problems. In the next section, we introduce the notion of a *convex function*, which plays an important role in our subsequent treatment of such problems.

21.2 CONVEX FUNCTIONS

We begin with a definition of the graph of a real-valued function.

Definition 21.1 The *graph* of $f : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^n$, is the set of points in $\Omega \times \mathbb{R} \subset \mathbb{R}^{n+1}$ given by

$$\{[\boldsymbol{x}, f(\boldsymbol{x})]^T : \boldsymbol{x} \in \Omega\}.$$

∎

We can visualize the graph of f as simply the set of points on a "plot" of $f(\boldsymbol{x})$ versus $\boldsymbol{x}$ (see Figure 21.4). We next define the "epigraph" of a real-valued function.

Definition 21.2 The *epigraph* of a function $f : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^n$, denoted epi$(f)$, is the set of points in $\Omega \times \mathbb{R}$ given by

$$\text{epi}(f) = \{[\boldsymbol{x}, \beta]^T : \boldsymbol{x} \in \Omega, \beta \in \mathbb{R}, \beta \geq f(\boldsymbol{x})\}.$$