$$A = \begin{bmatrix} -a & 1 & & \vdots & -b & & \vdots \\ & \ddots & \ddots & \vdots & & \ddots & \\ 0 & & -a & 1 & 0 & \cdots & -b \end{bmatrix}$$

$$b = \begin{bmatrix} ax_0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad z = [x_1, \ldots, x_N, u_1, \ldots, u_N]^T.$$

With these definitions, the problem reduces to the previously considered quadratic programming problem

$$\begin{array}{ll} \text{minimize} & \frac{1}{2} z^T Q z \\ \text{subject to} & Az = b, \end{array}$$

where Q is $2N \times 2N$, A is $N \times 2N$, and $b \in \mathbb{R}^N$. The solution is

$$z^* = Q^{-1} A^T (A Q^{-1} A^T)^{-1} b.$$

The first N components of z^* represent the optimal state signal in the interval $[1, N]$, whereas the second N components represent the optimal control signal.

In practice, the computation of the matrix inverses in the above formula for z^* may be too costly. There are other ways to tackle the problem by exploiting its special structure. This is the study of *optimal control* (see, e.g., [11], [15], [62], [63], or [71]). ■

The following example illustrates an application of the above discussion.

Example 19.10 *Credit-card holder dilemma.* Suppose we currently have a credit-card debt of \$10,000. Credit-card debts are subject to a monthly interest rate of 2%, and the account balance is increased by the interest amount every month. Each month, we have the option of reducing the account balance by contributing a payment to the account. Over the next 10 months, we plan to contribute a payment every month in such a way as to minimize the overall debt level while at the same time minimize the hardship of making monthly payments.

We solve our problem using the LQR framework as described in Example 19.9. Let the current time be 0, x_k the account balance at the end of month k, and u_k our payment in month k. We have

$$x_k = 1.02 x_{k-1} - u_k, \qquad k = 1, \ldots, 10,$$

that is, the account balance in a given month is equal to the account balance in the previous month plus the monthly interest on that balance minus our payment that month. Our optimization problem is then

$$\begin{array}{ll} \text{minimize} & \frac{1}{2} \sum_{i=1}^{10} (q x_i^2 + r u_i^2) \\ \text{subject to} & x_k = 1.02 x_{k-1} - u_k, \; k = 1, \ldots, 10, \; x_0 = 10000, \end{array}$$

which is an instance of the LQR problem. The parameters q and r reflect our priority in trading off between debt reduction and hardship in making payments. The more anxious we are to reduce our debt, the larger the value of q relative to r. On the other hand, the more reluctant we are to make payments, the larger the value of r relative to q.

The solution to the above problem is given by the formula derived in Example 19.9. In Figure 19.15, we plot the monthly account balances and payments over the next 10 months using $q = 1$ and $r = 10$. We can see here that our debt has been reduced to less than \$1,000 after 10 months, but with a first payment close to \$3,000. If we feel that a payment of \$3,000 is too high, then we can try to reduce this amount by increasing the value of r relative to q. However, going too far along these lines can lead to trouble. Indeed, if we use $q = 1$ and $r = 300$ (see Figure 19.16), although the monthly payments do not exceed \$400, the account balance is never reduced by much below \$10,000. In this case, the interest on the account balance eats up a significant portion of our monthly payments. In fact, our debt after 10 months will be higher than \$10,000. ■

19.1, 19.2, 19.3
19.7, 19.11

EXERCISES

19.1 Find local extremizers for the following optimization problems:

a.

$$\begin{array}{ll} \text{minimize} & x_1^2 + 2x_1x_2 + 3x_2^2 + 4x_1 + 5x_2 + 6x_3 \\ \text{subject to} & x_1 + 2x_2 = 3 \\ & 4x_1 + 5x_3 = 6; \end{array}$$

b.

$$\begin{array}{ll} \text{maximize} & 4x_1 + x_2^2 \\ \text{subject to} & x_1^2 + x_2^2 = 9; \end{array}$$

c.

$$\begin{array}{ll} \text{maximize} & x_1 x_2 \\ \text{subject to} & x_1^2 + 4x_2^2 = 1. \end{array}$$

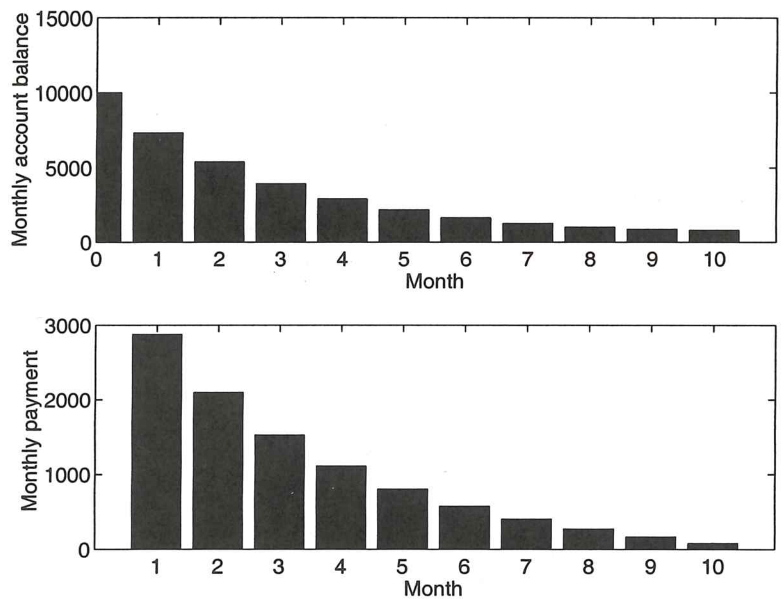

Figure 19.15 Plots for Example 19.10 with $q = 1$ and $r = 10$

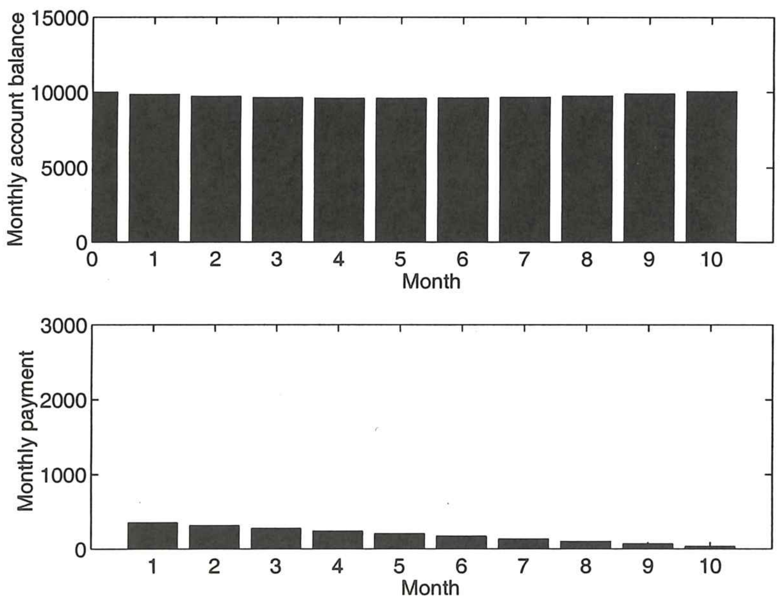

Figure 19.16 Plots for Example 19.10 with $q = 1$ and $r = 300$

19.2 Consider the problem

$$\begin{array}{ll} \text{minimize} & f(\boldsymbol{x}) \\ \text{subject to} & h(\boldsymbol{x}) = 0, \end{array}$$

where $f : \mathbb{R}^2 \to \mathbb{R}$, $h : \mathbb{R}^2 \to \mathbb{R}$, and $\nabla f(\boldsymbol{x}) = [x_1, x_1 + 4]^T$. Suppose that $\boldsymbol{x}^*$ is an optimal solution, and $\nabla h(\boldsymbol{x}^*) = [1, 4]^T$. Find $\nabla f(\boldsymbol{x}^*)$.

19.3 Consider the problem

$$\begin{array}{ll} \text{minimize} & \|\boldsymbol{x} - \boldsymbol{x}_0\|^2 \\ \text{subject to} & \|\boldsymbol{x}\|^2 = 9, \end{array}$$

where $\boldsymbol{x}_0 = [1, \sqrt{3}]^T$.

a. Find all points satisfying the Lagrange condition for the problem.

b. Using second-order conditions, determine whether or not each of the points in part a are local minimizers.

19.4 We wish to construct a closed box with minimum surface area that encloses a volume of V cubic feet, where $V > 0$.

a. Let a, b, and c denote the dimensions of the box with minimum surface area (with volume V). Derive the Lagrange condition that must be satisfied by a, b, and c.

b. What does it mean for a point $\boldsymbol{x}^*$ to be a *regular* point in this problem? Is the point $\boldsymbol{x}^* = [a, b, c]^T$ a regular point?

c. Find a, b, and c.

d. Does the point $\boldsymbol{x}^* = [a, b, c]^T$ found in part c satisfy the second-order sufficient condition?

19.5 Find local extremizers of

a. $f(x_1, x_2, x_3) = x_1^2 + 3x_2^2 + x_3$ subject to $x_1^2 + x_2^2 + x_3^2 = 16$;

b. $f(x_1, x_2) = x_1^2 + x_2^2$ subject to $3x_1^2 + 4x_1x_2 + 6x_2^2 = 140$.

19.6 Consider the problem

$$\begin{array}{lll} \text{minimize} & 2x_1 + 3x_2 - 4, & x_1, x_2 \in \mathbb{R} \\ \text{subject to} & x_1x_2 = 6. & \end{array}$$

a. Use Lagrange's theorem to find all possible local minimizers and maximizers.

b. Use the second-order sufficient conditions to specify which points are strict local minimizers and which are strict local maximizers.

c. Are the points in part b global minimizers or maximizers? Explain.

19.7 Find all solutions to the problem

$$\begin{array}{ll} \text{maximize} & \boldsymbol{x}^T \begin{bmatrix} 3 & 4 \\ 0 & 3 \end{bmatrix} \boldsymbol{x} \\ \text{subject to} & \|\boldsymbol{x}\|^2 = 1. \end{array}$$

19.8 Consider a matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n}$. Define the *induced 2-norm* of $\boldsymbol{A}$, denoted $\|\boldsymbol{A}\|_2$, to be the number

$$\|\boldsymbol{A}\|_2 = \max\{\|\boldsymbol{A}\boldsymbol{x}\| : \boldsymbol{x} \in \mathbb{R}^n,\ \|\boldsymbol{x}\| = 1\},$$

where the norm $\|\cdot\|$ on the right-hand side above is the usual Euclidean norm.

Suppose the eigenvalues of $\boldsymbol{A}^T\boldsymbol{A}$ are $\lambda_1, \ldots, \lambda_n$ (ordered from largest to smallest). Use Lagrange's theorem to express $\|\boldsymbol{A}\|_2$ in terms of the above eigenvalues (cf. Theorem 3.8).

19.9 Let $\boldsymbol{P} = \boldsymbol{P}^T$ be a positive definite matrix. Show that any point $\boldsymbol{x}$ satisfying $1 - \boldsymbol{x}^T\boldsymbol{P}\boldsymbol{x} = 0$ is a regular point.

19.10 Consider the problem:

$$\begin{array}{lll} \text{maximize} & ax_1 + bx_2, & x_1, x_2 \in \mathbb{R} \\ \text{subject to} & x_1^2 + x_2^2 = 2, & \end{array}$$

where $a, b \in \mathbb{R}$. Show that if $[1, 1]^T$ is a solution to the problem, then $a = b$.

19.11 Consider the problem:

$$\begin{array}{lll} \text{minimize} & x_1x_2 - 2x_1, & x_1, x_2 \in \mathbb{R} \\ \text{subject to} & x_1^2 - x_2^2 = 0. & \end{array}$$

a. Apply Lagrange's theorem directly to the problem to show that if a solution exists, it must be either $[1, 1]^T$ or $[-1, 1]^T$.

b. Use the second-order necessary conditions to show that $[-1, 1]^T$ cannot possibly be the solution.

c. Use the second-order sufficient conditions to show that $[1, 1]^T$ is a strict local minimizer.

19.12 Let $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, $m \leq n$, rank $\boldsymbol{A} = m$, and $\boldsymbol{x}_0 \in \mathbb{R}^n$. Let $\boldsymbol{x}^*$ be the point on the nullspace of $\boldsymbol{A}$ that is closest to $\boldsymbol{x}_0$ (in the sense of Euclidean norm).

a. Show that $\boldsymbol{x}^*$ is orthogonal to $\boldsymbol{x}^* - \boldsymbol{x}_0$.

b. Find a formula for $\boldsymbol{x}^*$ in terms of $\boldsymbol{A}$ and $\boldsymbol{x}_0$.

19.13 Consider the *quadratic programming* problem

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}\boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x} \\ \text{subject to} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \end{array}$$

where $\boldsymbol{Q} = \boldsymbol{Q}^T > 0$, $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, $m < n$, and rank $\boldsymbol{A} = m$. Use the Lagrange condition to derive a closed-form solution to the problem.

19.14 Let $\boldsymbol{L}$ be an $n \times n$ real symmetric matrix, and let $\mathcal{M}$ be a subspace of $\mathbb{R}^n$ with dimension $m < n$. Let $\{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_m\} \subset \mathbb{R}^n$ be a basis for $\mathcal{M}$, and let $\boldsymbol{B}$ be the $n \times m$ matrix with $\boldsymbol{b}_i$ as the ith column. Let $\boldsymbol{L}_{\mathcal{M}}$ be the $m \times m$ matrix defined by $\boldsymbol{L}_{\mathcal{M}} = \boldsymbol{B}^T\boldsymbol{L}\boldsymbol{B}$. Show that $\boldsymbol{L}$ is positive semidefinite (definite) on $\mathcal{M}$ if and only if $\boldsymbol{L}_{\mathcal{M}}$ is positive semidefinite (definite).

Note: This result is useful for checking that the Hessian of the Lagrangian function at a point is positive definite on the tangent space at that point.

19.15 Consider the sequence $\{x_k\}$, $x_k \in \mathbb{R}$, generated by the recursion

$$x_{k+1} = ax_k + bu_k, \quad k \geq 0 \qquad (a, b \in \mathbb{R},\ a, b \neq 0),$$

where $u_0, u_1, u_2, \ldots$ is a sequence of "control inputs," and the initial condition $x_0 \neq 0$ is given. The above recursion is also called a *discrete-time linear system*. We wish to find values of control inputs u_0 and u_1 such that $x_2 = 0$, and the average input energy $(u_0^2 + u_1^2)/2$ is minimized. Denote the optimal inputs by u_0^* and u_1^*.

a. Find expressions for u_0^* and u_1^* in terms of a, b, and x_0.

b. Use the second-order sufficient conditions to show that the point $\boldsymbol{u}^* = [u_0^*, u_1^*]^T$ in part a is a strict local minimizer.

19.16 Consider the discrete-time linear system $x_k = 2x_{k-1} + u_k$, $k \geq 1$, with $x_0 = 1$. Find the values of the control inputs u_1 and u_2 to minimize

$$x_2^2 + \frac{1}{2}u_1^2 + \frac{1}{3}u_2^2.$$