Final Exam for Introduction to Optimization

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General Guideline:

1. **There are 12 questions in this exam. Please do all questions. The total score is 360. Do your best and good luck.**
2. **Your answers can be written either in English or in Chinese.**
3. **Please give proper definition for the symbols you use.**
4. **Write all answers on the blank place immediately following the questions. If there is no enough space, continue your answers on the back of the sheet with proper indications.**
5. **Partial grade may be given. Write down any derivation to demonstrate your knowledge in case your answer is incorrect.**
6. **If the problem is infeasible, please state why it is infeasible.**
7. **Please return your answers in a PDF file to me by email sfsu@mail.ntust.edu.tw before 5/30 12:00.**

Your Name: __________________________

Student ID: __________________________

Problem Number	Score
I. (15pts)	
II. (50pts)	
III. (40pts)	
IV. (20pts)	
V. (30pts)	
VI. (20pts)	
VII. (25pts)	
VIII. (20pts)	
IX. (20 pts)	
X. (50 pts)	
XI. (50pts)	
XII. (20 pts)	
Total: (360pts)	

I. (15 points) Prove that $x^* = A^T(AA^T)^{-1}b$ is the **unique** solution that minimizes $||x||$ subject to $Ax=b$.

II. Let $A_0 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}$, $b_0 = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$, $A_1 = [1 \ \ 2]$, $b_1 = [4]$, and $A_2 = [0 \ \ 1]$, $b_2 = [2]$.

(a) (10 points) Compute the vector $x^{(0)}$ that minimizes $\|A_0 x - b_0\|^2$.

(b) (20 points) Use the RLS algorithm to find $x^{(2)}$ that minimizes $\left\| \begin{bmatrix} A_0 \\ A_1 \\ A_2 \end{bmatrix} x - \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} \right\|^2$.

(c) (20 points) Use the RLS algorithm for the problem by considering one datum at a time; i.e., the training data set are {(1, 0; 1); (1, 1; 3); (2, 1; 4); (1, 2; 4); (0, 1; 2)}.

III. (a) (20 points) Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ and $b = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}$. Show 3 iterations of Kaczmarz's algorithm with $\mu = 1$ and $x^{(0)} = \mathbf{0}$.

(b) (20 points) Repeat the above with $\mu = 0.8$ and $x^{(0)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

IV. Consider the problem of maximizing $3x_1 + x_2$ subject to $x_1 + 2x_2 \leq 40$, $2x_1 + x_2 \leq 20$, $x_1 + x_2 \leq 12$ and $x_1, x_2 \geq 0$.

(a) (5 points) Please convert it into a standard linear programming problem.

(b) (15 points) Use the simplex method to find the solution of the problem.

V. Consider the problem of maximizing $x_1 - 2x_2$ subject to $x_1 \leq 0$ and $x_2 \geq 1$.

(a) (5 points) Please convert it into a standard linear programming problem.

(b) (25 points) Use the two-phase simplex method to find the solution of the problem.

VI. (20 points) Let x_0 satisfy $Ax=b$ and $x \geq 0$, where $A \in \mathbb{R}^{m\times n}$ and $m \leq n$. Please prove that there exists a basic feasible solution for $Ax=b$ and $x \geq 0$.

VII. (25 points) Consider the linear program maximize $x_1 + 2x_2$ subject to $x_1 + 2x_2 \geq 4$, $2x_1 + x_2 \geq 3$, and $x_1, x_2 \geq 0$. Find the dual to this problem and solve the dual problem.

VIII.(20 points) Find local extremizers for the problem: Maximize $x_1 x_2$ subject to $x_1^2 + 4x_2^2 = 2$.

IX. (20 points) Find the solution for the problem: minimize $x_1^2 + x_1x_2 - 4x_1 - 2x_2$ subject to $x_1^2 + x_2 \le 1,\ 2x_1 - x_2 \le 1$.

X. Short Answers. (*5 pts each*)

1. What is the major idea for the gradient method?

ANS:

2. Consider a quadratic function $f: \mathbb{R}^n \to \mathbb{R}$. If Newton's method is applied, how many iterations does it need to reach the minimizer?

ANS:

3. Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $m \le n$, rank $A = m$, and $x_0 \in \mathbb{R}^n$. Please give the minimizer for $\|x - x_0\|$ subject to $AX = b$.

ANS:

4. In Kaczmarz's algorithm, what will happen if the initial condition $x^{(0)}$ is not zero?

ANS:

5. What is a basic feasible solution for a standard linear programming problem?

ANS:

6. What are the canonical augmented matrices for a given linear programming problem?

ANS:

8. Given a primal linear programming problem: Minimizer $c^T x$, subject to $Ax \ge b$ and $x \ge 0$, please specify its dual problem.

ANS:

9. What advantage can the two-phase simplex approach provide?

ANS:

9. For a standard constrained optimization problem, what is the idea for the Lagrange multiplier?

ANS:

10. For a standard constrained optimization problem, why need to add $\mu^* \geq 0$ in the Karush-Kuhn-Tuker (KKT) conditions?

ANS:

XI. True/False. Be sure to justify your answer. (*5 pts each*)

1. Newton's method is always good as long as the calculation of the inverse of the corresponding Hessian matrix has no problem.

ANS:_____ Why:

2. By using the conjugate direction algorithm, the search can converge to the minimizer in *n* steps for any starting point.

ANS:_____ Why:

3. A basic solution for a LP is always a feasible solution.

ANS:_____ Why:

4. For a standard LP problem, if there exists a basic feasible solution, then the optimal solution exists and the optimal solution also is a basic feasible solution for that LP problem.

ANS:_____ Why:

5. A degenerate basic solution for a LP problem is always a feasible solution.

ANS:_____ Why:

6. The two phase simplex method is to separate a LP problem into two solvable standard LP sub-problems.

ANS:_____ Why:

7. The revised simplex approach is used to find the solution with less calculations than the traditional simplex method does.

ANS:_____ Why:

8. A vector in the normal space of the constraint surface of a constrained optimization problem can be viewed as the feasible direction on this surface.

ANS:_____ Why:

9. A point whose gradient is parallel to the tangent space of the constraint surface of a constrained optimization problem, this point is possible an extreme point for the problem.

ANS:_____ Why:

10. The Karush-Kuhn-Tuker (KKT) approach is an extended version of the Lagrange theorem by considering either interior points or boundary points to form the KKT conditions.

ANS:_____ Why:

XII. Please give your comments for this course. (20 points)
(Note that any comments are fine and can be granted for 20 points.)

— END —

Have a nice vacation and a happy summer vacation.